Math 142 Lecture 16 Notes

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1 The Brouwer Fixed Point Theorem and Introduction to Manifolds

1.1 The Brouwer fixed point theorem

One of the reasons why people study algebraic topology is that it can tell us things unrelated to topology itself. Here is one such theorem.

Theorem 1.1 (Brouwer)**.** *If $f : B^n \to B^n$ is continuous, then f has a fixed point; i.e. $\exists x \in B^n$ such that $f(x) = x$.*

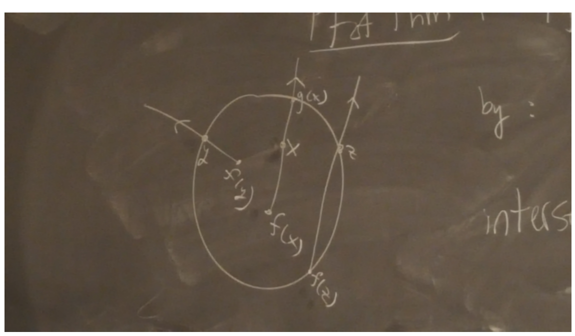
To prove this, we need the following proposition:

Proposition 1.1. *There does not exist a continuous map $f : B^n \to S^{n-1}$ such that $f(x) = x$ for all $x \in \partial B^n = S^{n-1}$.*

Proof. We did the proof of the case $n = 2$ on homeowork 6, and we will show this for $n > 3$ later. For $n = 1$, if $f : [0, 1] \to \{0, 1\}$ is continuous, then $f([0, 1])$ is connected, so f is not surjective. ∎

Now let's prove the fixed point theorem.

Proof. (Brouwer fixed point) Proceed by contradiction. If $f(x) \neq x$ for all $x \in B^n$, define $g : B^n \to S^{n-1}$ by drawing a ray from $f(x)$ to x and defining $g(x)$ to be the intersection of the ray with the sphere (that is not equal to $f(x)$).



Note that $g(x) = x$ for all $x \in S^{n-1}$. Check for yourself that g is a continuous function (try coming up with a formula for it). But such a g cannot exist by the previous proposition. $\square$

1.2 Introduction to Manifolds

Definition 1.1. A topological space X is *second-countable* if there exists a countable base for its topology.

Definition 1.2. A *manifold of dimension* n (or an n*-manifold*) is a topological space X such that:

1. X is Hausdorff.
2. X is second-countable.
3. $\forall x \in X$, there is an open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^n$.

The pair (U_x, ϕ) is called a *chart*.

Remark 1.1. Why do we have the first two conditions? The second-countable condition excludes "weird" spaces like the "long line." The Hausdorff condition excludes spaces like the "line with 2 origins." This is $X = \mathbb{R} \cup \{0'\}$, where a set $U \subseteq X$ is open if

- $U \subseteq \mathbb{R}$, and U is open in the usual topology on $\mathbb{R}$.
- $U = (U' \setminus \{0\}) \cup \{0'\}$, where $U' \subseteq \mathbb{R}$ is open in the usual topology on $\mathbb{R}$, and $0 \in U'$.

This is second-countable, and around $x = 0'$, $(x-\varepsilon, x+\varepsilon) \cong \mathbb{R}$ and $((-\varepsilon, \varepsilon)\setminus\{0\}) \cup \{0\} \cong \mathbb{R}$, so it satisfies the 3rd condition of being a manifold.

Example 1.1. $\mathbb{R}^n$ is an n-manifold.

Example 1.2. S^n is an n-manifold. If $x \in S^n$, then $S^n \setminus \{-x\}$ is an open neighborhood of x that is homeomorphic to $\mathbb{R}^n$.

Example 1.3. If X is an n-manifold, and $U \subseteq X$ is an open subspace, then U is an n-manifold.

Proposition 1.2. *If X is an n-manifold, and Y is an m-manifold, then $X \times Y$ is an $(n+m)$-manifold.*

Example 1.4. T^n is an n-manifold.

Proposition 1.3. *If X is an n-manifold, and G acts "nicely" on X, then X/G is an n-manifold.*

Proof. Given $x \in X$, let U_x be an open neighborhood of x such that $f_g(U_x) \cap U_x = \varnothing$ $\forall g \neq 0$. Then if $\pi : X \to X/G$ is the natural projection map, then $\pi|_{U_x} : U_x \to \pi(U_x)$ is a homeomorphism. The rest of the proof may be assigned for homework. $\square$

Example 1.5. $\mathbb{R}P^n$ is an n-manifold.

Example 1.6. $L(p, q)$ is a 3-manifold.

Example 1.7. The Klein bottle and the torus are 2-manifolds.

Definition 1.3. An *n-manifold with boundary* is a topological space X such that

1. X is Hausdorff.

2. X is second-countable.

3. $\forall x \in X$, there exists an open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^n$ or $\phi : U_x \to \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

The pair (U_x, ϕ) is still called a *chart*. The *interior* of X is

$$\text{int}(X) = \{x \in X : \exists \text{ chart } (U_x, \phi) \text{ s.t. } U_x \cong \mathbb{R}^n\}.$$

The *boundary* of X is

$$\partial X = X \setminus \text{int}(X) = \{x \in X : \exists \text{ chart } (U_x, \phi) \text{ s.t. } \phi(x) \in \{x_n = 0\}\}.$$

Remark 1.2. Often, authors will talk about manifolds with boundary just as "manifolds." You should always check to see which terminology is being used in whatever you are reading. There have been published results that are incorrect because they cited a result from literature without checking to make sure that the source was using the correct definition of "manifold" for their usage.

Definition 1.4. A manifold X (with boundary) is called *closed* if it is compact and $\partial X = \varnothing$.

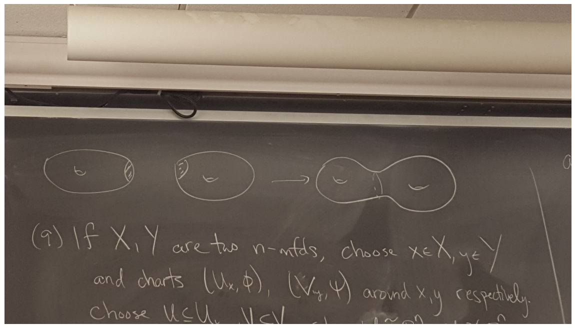
Proposition 1.4. *If X is an n-manifold with boundary, then ∂X is an $(n-1)$-manifold.*

Example 1.8. B^n is an n-manifold with boundary, and $\partial B^n = S^{n-1}$.

Proposition 1.5. *If X, Y are two n-manifolds with boundary, and $f : \partial Y \to \partial X$ is a homeomorphism, then $X \cup_f Y$ is an n-manifold.*

Example 1.9. If X and Y are two n-manifolds, choose $x \in X$, $y \in Y$ and charts (U_x, ϕ), (V_y, ϕ). Choose $U \subseteq U_x$ and $V \subseteq V_y$ such that $U \cong_\phi B^n$ and $V \cong_\psi B^n$. Then $X' = X \setminus \text{int}(U)$ and $Y' \setminus \text{int}(V)$ are n manifolds with boundary, and $\partial X', \partial Y' \cong S^{n-1}$. Choose a homeomorphism $f : \partial Y' \to \partial X'$. Then the *connected sum* of X and Y is $X' \# Y' :=$

$X' \cup_f Y'$. (If X and Y are path-connected, then different choices of x, y, U, V, and f give homeomorphic manifolds.)



Theorem 1.2. *(Generalized Poincare ~~Conjecture~~ Theorem for topological manifolds) If X is a closed, connected n-manifold, and X is homotopy equivalent to S^n, then X is homeomorphic to S^n.*

The $n = 1, 2$ cases are "classical," and we will prove this by classifying such 1 and 2-manifolds. The $n = 3$ case was proved by Perelman in 2003, which won him a Fields medal and other prizes, all of which he rejected. The $n = 4$ case was proved by Freedman in 1982, and the $n \geq 5$ case was proven by Smale in 1960-1961.